REPRESENTATIONS OF SOME LATTICES INTO THE GROUP OF ANALYTIC DIFFEOMORPHISMS OF THE SPHERE $\mathbb{S}^2$

by

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Abstract. — In [**10**] it is proved that any morphism from a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$ to the group of analytic diffeomorphisms of $\mathbb{S}^2$ has a finite image as soon as $n \geq 5$. The case $n = 4$ is also claimed to follow along the same arguments; in fact this is not straightforward and this case indeed needs a modification of the argument. In this paper we recall the strategy for $n \geq 5$ and then focus on the case $n = 4$.
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1. Introduction

After the works of Margulis ([**14**, **19**]) on the linear representations of lattices of simple, real Lie groups with $\mathbb{R}$-rank larger than 1, some authors, like Zimmer, suggest to study the actions of lattices on compact manifolds ([**20**, **21**, **22**, **23**]). One of the main conjectures of this program is the following: let us consider a connected, simple, real Lie group G and let Γ be a lattice of G of $\mathbb{R}$-rank larger than 1. If there exists a morphism of infinite image from Γ to the group of diffeomorphisms of a compact manifold M, then the $\mathbb{R}$-rank of G is bounded by the dimension of M. There are a lot of contributions in that direction ([**3**, **4**, **5**, **7**, **8**, **9**, **10**, **11**, **16**, **17**]). In this article we will focus on the embeddings of subgroups of finite index of $\mathrm{SL}(n,\mathbb{Z})$ into the group $\mathrm{Diff}^\omega(\mathbb{S}^2)$ of real analytic diffeomorphisms of $\mathbb{S}^2$ (*see* [**10**]).

The article is organized as follows. First of all we will recall the strategy of [**10**]: the study of the nilpotent subgroups of $\mathrm{Diff}^\omega(\mathbb{S}^2)$ implies that such subgroups are metabelian. But subgroups of finite index of $\mathrm{SL}(n,\mathbb{Z})$, for $n \geq 5$, contain nilpotent subgroups of length $n-1$ of finite index which are not metabelian; as a consequence Ghys gets the following statement.

Theorem A **([10]).** — *Let Γ be a subgroup of finite index of* $\mathrm{SL}(n,\mathbb{Z})$*. As soon as $n \geq 5$ there is no embedding of Γ into* $\mathrm{Diff}^\omega(\mathbb{S}^2)$.

To study nilpotent subgroups of $\mathrm{Diff}^\omega(\mathbb{S}^2)$ one has to study nilpotent subgroups of $\mathrm{Diff}^\omega_+(\mathbb{S}^1)$ (*see* §2) and then nilpotent subgroups of the group of formal diffeomorphisms of $\mathbb{C}^2$ (*see* §3). The last section is devoted to establish the following result.

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Theorem B. — *Let* Γ *be a subgroup of finite index of* $\mathrm{SL}(n,\mathbb{Z})$*. As soon as* $n\geq 4$ *there is no embedding of* Γ *into* $\mathrm{Diff}^\omega(\mathbb{S}^2)$.

The proof relies on the characterization, up to isomorphism, of nilpotent subalgebras of length 3 of the algebra of formal vector fields of $\mathbb{C}^2$ which vanish at the origin.

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2. Nilpotent subgroups of the group of analytic diffeomorphisms of $\mathbb{S}^1$

Let G be a group; let us set

$$\mathrm{G}^{(0)}=\mathrm{G} \qquad \& \qquad \mathrm{G}^{(i)}=[\mathrm{G},\mathrm{G}^{(i-1)}]\ \ \forall i\geq 1.$$

The group G is *nilpotent* if there exists an integer n such that $\mathrm{G}^{(n)}=\{\mathrm{id}\}$; the *length of nilpotence* of G is the smallest integer k such that $\mathrm{G}^{(k)}=\{\mathrm{id}\}$.

Set

$$\mathrm{G}_{(0)}=\mathrm{G} \qquad \& \qquad \mathrm{G}_{(i)}=[\mathrm{G}_{(i-1)},\mathrm{G}_{(i-1)}]\ \ \forall i\geq 1.$$

The group G is *solvable* if $\mathrm{G}_{(n)}=\{\mathrm{id}\}$ for a certain n; the *length of solvability* of G is the smallest integer k such that $\mathrm{G}_{(k)}=\{\mathrm{id}\}$.

We say that the group G (resp. algebra $\mathfrak{g}$) is *metabelian* if $[\mathrm{G},\mathrm{G}]$ (resp. $[\mathfrak{g},\mathfrak{g}]$) is abelian.

Proposition 2.1 **([10])**. — *Any nilpotent subgroup of* $\mathrm{Diff}^\omega_+(\mathbb{S}^1)$ *is abelian.*

Proof. — Let G be a nilpotent subgroup of $\mathrm{Diff}^\omega_+(\mathbb{S}^1)$. Assume that G is not abelian; it thus contains a Heisenberg group

$$\langle f,g,h\,|\,[f,g]=h,\,[f,h]=[g,h]=\mathrm{id}\rangle.$$

The application “rotation number“

$$\mathrm{Diff}^\omega_+(\mathbb{S}^1)\to\mathbb{R}/\mathbb{Z}, \qquad \psi\mapsto\lim_{n\to+\infty}\frac{\psi^n(x)-x}{n}$$

is not a morphism but its restriction to a solvable subgroup is ([1]). Thus the rotation number of h is zero and the set $\mathrm{Fix}(h)$ of fixed points of h is non-empty and finite. Considering some iterates of f and g instead of f and g one can assume that f and g fix any point of $\mathrm{Fix}(h)$. The set of fixed points of a non trivial element of $\langle f,g\rangle$ is finite and invariant by h so the action of $\langle f,g\rangle$ is free [(1)] on each component of $\mathbb{S}^1\setminus\mathrm{Fix}(h)$. But the action of a free group on $\mathbb{R}$ is abelian: contradiction. □

1. The stabilizer of every point is trivial, *i.e.* the action of a non trivial element of $\langle f,g\rangle$ has no fixed point.

3. Nilpotent subgroups of the group of formal diffeomorphisms of $\mathbb{C}^2$

Let us denote $\widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$ the group of formal diffeomorphisms of $\mathbb{C}^2$, *i.e.* the formal completion of the group of germs of holomorphic diffeomorphisms at 0. For any i let Diff_i be the quotient of $\widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$ by the normal subgroups of formal diffeomorphisms tangent to the identity with multiplicity i; it can be viewed as the set of jets of diffeomorphisms at order i with the law of composition with truncation at order i. Note that Diff_i is a complex linear algebraic group. One can see $\widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$ as the projective limit of the Diff_i's: $\widehat{\mathrm{Diff}}(\mathbb{C}^2,0)=\lim\limits_{\leftarrow}\mathrm{Diff}_i$. Let us denote by $\widehat{\chi}(\mathbb{C}^2,0)$ the algebra of formal vector fields in $\mathbb{C}^2$ vanishing at 0. One can define the set χ_i of the i-th jets of vector fields; one has $\lim\limits_{\leftarrow}\chi_i=\widehat{\chi}(\mathbb{C}^2,0)$.

Let $\widehat{O}(\mathbb{C}^2)$ be the ring of formal series in two variables and let $\widehat{K}(\mathbb{C}^2)$ be its fraction field; O_i is the set of elements of $\widehat{O}(\mathbb{C}^2)$ truncated at order i.

The family $\left(\exp_i\colon \chi_i\to\mathrm{Diff}_i\right)_i$ is filtered, *i.e.* compatible with the truncation. We then define the exponential application as follows: $\exp=\lim\limits_{\leftarrow}\exp_i\colon\widehat{\chi}(\mathbb{C}^2,0)\to\widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$.

As in the classical case, if X belongs to $\widehat{\chi}(\mathbb{C}^2,0)$, then $\exp(X)$ can be seen as the "flow at time 1" of X. Indeed an element X_i of χ_i can be seen as a derivation of O_i; so it can be written S_i+N_i where S_i and N_i are two semi-simple, resp. nilpotent derivations which commute. Taking the limit, one gets $X=S+N$ where S is a semi-simple vector field and N a nilpotent one and $[S,N]=\mathrm{id}$ (*see* [15]). A semi-simple vector field is a formal vector field conjugate to a diagonal linear vector field which is complete. A vector field is nilpotent if and only if its linear part is; let us remark that the usual flow φ_t of a nilpotent vector field is polynomial in t

$$\varphi_t(x)=\sum_I P_I(t)x^I, \qquad\qquad P_I\in(\mathbb{C}[t])^2$$

so $\varphi_1(x)$ is well defined. As a consequence $\exp(tX)=\exp(tS)\exp(tN)$ is well defined for $t=1$. Note that the Jordan decomposition is purely formal: if X is holomorphic, S and N are not necessary holomorphic.

***Proposition 3.1* ([10]).** — *Any nilpotent subalgebra of $\widehat{\chi}(\mathbb{C}^2,0)$ is metabelian.*

Proof. — Let $\mathfrak{l}$ be a nilpotent subalgebra of $\widehat{\chi}(\mathbb{C}^2,0)$ and let $Z(\mathfrak{l})$ be its center. Since $\widehat{\chi}(\mathbb{C}^2,0)\otimes\widehat{K}(\mathbb{C}^2)$ is a vector space of dimension 2 over $\widehat{K}(\mathbb{C}^2)$ one has the following alternative:

- the dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2,0)\otimes\widehat{K}(\mathbb{C}^2)$ is 1;
- the dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2,0)\otimes\widehat{K}(\mathbb{C}^2)$ is 2.

Let us study these different cases.

Under the first assumption there exists an element X of $Z(\mathfrak{l})$ having the following property: any vector field of $Z(\mathfrak{l})$ can be written uX with u in $\widehat{K}(\mathbb{C}^2)$. Let us consider the subalgebra $\mathfrak{g}$ of $\mathfrak{l}$ given by

$$\mathfrak{g}=\left\{\widetilde{X}\in\mathfrak{l}\,|\,\exists u\in\widehat{K}(\mathbb{C}^2),\,\widetilde{X}=uX\right\}.$$

Since X belongs to $Z(\mathfrak{l})$, the algebra $\mathfrak{g}$ is abelian; it is also an ideal of $\mathfrak{l}$. Let us assume that $\mathfrak{l}$ is not abelian: let Y be an element of $\mathfrak{l}$ whose projection on $\mathfrak{l}/\mathfrak{g}$ is non trivial and central. Any vector field of $\mathfrak{l}$ can be written as $uX+vY$ with u, v in $\widehat{K}(\mathbb{C}^2)$. As X belongs to $Z(\mathfrak{l})$ and Y is central modulo $\mathfrak{g}$ one has

$$X(u)=X(v)=Y(v)=0.$$

The vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ being some linear combinations of X and Y with coefficients in $\widehat{K}(\mathbb{C}^2,0)$, the partial derivatives of v are zero so v is a constant. Therefore $[\mathfrak{l},\mathfrak{l}]\subset\mathfrak{g}$; but $\mathfrak{g}$ is abelian so $\mathfrak{l}$ is metabelian.

In the second case $Z(\mathfrak{l})$ has two elements X and Y which are linearly independent on $\widehat{K}(\mathbb{C}^2)$. Any vector field of $\mathfrak{l}$ can be written as $uX+vY$ with u and v in $\widehat{K}(\mathbb{C}^2)$. Since X and Y belong to $Z(\mathfrak{l})$ one has

$$X(u) = X(v) = Y(u) = Y(v) = 0.$$

As a consequence u and v are constant, *i.e.* $\mathfrak{l} \subset \{uX+vY \,|\, u, v \in \mathbb{C}\}$; in particular $\mathfrak{l}$ is abelian. □

Proposition 3.2 **([10]).** — *Any nilpotent subgroup of* $\widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$ *is metabelian.*

Proof. — Let G be a nilpotent subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$ of length k. Let us denote G_i the projection of G on Diff_i. The Zariski closure $\overline{\mathrm{G}_i}$ of G_i in Diff_i is an algebraic nilpotent subgroup of length k. It is sufficient to prove that $\overline{\mathrm{G}_i}$ is metabelian.

Since $\overline{\mathrm{G}_i}$ is a complex algebraic subgroup it is the direct product of the subgroup $\overline{\mathrm{G}_{i,u}}$ of its unipotent elements and the subgroup $\overline{\mathrm{G}_{i,s}}$ of its semi-simple elements (*see for example* [2]).

An element of Diff_i is unipotent if and only if its linear part, which is in $\mathrm{GL}(2,\mathbb{C})$, is; so $\overline{\mathrm{G}_{i,s}}$ projects injectively onto a nilpotent subgroup of $\mathrm{GL}(2,\mathbb{C})$. Therefore $\overline{\mathrm{G}_{i,s}}$ is abelian.

Let us now consider $\overline{\mathrm{G}_{i,u}}$; this group is the exponential of a nilpotent Lie algebra $\mathfrak{l}_i$ of χ_i of length k. Taking the limit one thus obtains the existence of a nilpotent subalgebra $\mathfrak{l}$ of $\widehat{\chi}(\mathbb{C}^2,0)$ of length k such that $\exp(\mathfrak{l})$ projects onto $\overline{\mathrm{G}_{i,u}}$ for any i. According to Proposition 3.1 the subalgebra $\mathfrak{l}$ and thus $\overline{\mathrm{G}_{i,u}}$ are metabelian. □

4. Nilpotent subgroups of the group of analytic diffeomorphisms of $\mathbb{S}^2$

Proposition 4.1 **([10]).** — *Any nilpotent subgroup of* $\mathrm{Diff}^\omega(\mathbb{S}^2)$ *has a finite orbit.*

Proof. — Let G be a nilpotent subgroup of $\mathrm{Diff}^\omega(\mathbb{S}^2)$; up to finite index one can assume that the elements of G preserve the orientation. Let ϕ be a non trivial element of G which commutes with G. Let $\mathrm{Fix}(\phi)$ be the set of fixed points of ϕ; it is a non empty analytic subspace of $\mathbb{S}^2$ invariant by G. If p is an isolated fixed point of ϕ, then the orbit of p under the action of G is finite. So it is sufficient to study the case where $\mathrm{Fix}(\phi)$ only contains curves; there are thus two possibilities:

- $\mathrm{Fix}(\phi)$ is a singular analytic curve whose set of singular points is a finite orbit for G;
- $\mathrm{Fix}(\phi)$ is a smooth analytic curve, not necessary connected. One of the connected component of $\mathbb{S}^2\setminus\mathrm{Fix}(\phi)$ is a disk denoted $\mathbb{D}$. Any subgroup Γ of finite index of G which contains ϕ fixes $\mathbb{D}$. Let us consider an element γ of Γ and a fixed point m of γ which is in $\overline{\mathbb{D}}$. By construction ϕ has no fixed point in $\mathbb{D}$ so according to the Brouwer Theorem $(\phi^k(m))_k$ has a limit point on the boundary $\partial\mathbb{D}$ of $\overline{\mathbb{D}}$. Therefore γ has at least one fixed point on $\partial\mathbb{D}$. The group Γ thus acts on the circle $\partial\mathbb{D}$ and any of its elements has a fixed point on $\mathbb{D}$. Then Γ has a fixed point on $\partial\mathbb{D}$ (Proposition 2.1).

□

Theorem 4.2 **([10]).** — *Any nilpotent subgroup of* $\mathrm{Diff}^\omega(\mathbb{S}^2)$ *is metabelian.*

Proof. — Let G be a nilpotent subgroup of $\mathrm{Diff}^\omega(\mathbb{S}^2)$ and let Γ be a subgroup of finite index of G having a fixed point m (such a subgroup exists according to Proposition 4.1). One can embed Γ into $\widehat{\mathrm{Diff}}(\mathbb{R}^2,0)$, and so into $\widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$, by considering the jets of infinite order of elements of Γ in m. According to Proposition 3.2 the group Γ is metabelian.

One can assume that G is a finitely generated group.

Let us first assume that G has no element of finite order. Then G is a cocompact lattice of the nilpotent, simply connected Lie group $\mathrm{G}\otimes\mathbb{R}$ (*see* [18]). The group G is metabelian if and only if $\mathrm{G}\otimes\mathbb{R}$ is; but Γ is metabelian so $\mathrm{G}\otimes\mathbb{R}$ also.

Finally let us consider the case where G has at least one element of finite order. The set of such elements is a normal subgroup of G which thus intersects non trivially the center $Z(\mathrm{G})$ of G. Let us consider a non trivial element ϕ of $Z(\mathrm{G})$ which has finite order. Let us recall that a finite group of diffeomorphisms of the sphere is conjugate to a group of isometries. Denote by G^+ the subgroup of elements of G which preserve the orientation. It is thus sufficient to prove that G^+ is metabelian; indeed if ϕ does not preserve the orientation, ϕ has order 2 and $\mathrm{G}=\mathbb{Z}/2\mathbb{Z}\times\mathrm{G}^+$. So let us assume that ϕ preserves the orientation; ϕ is conjugate to a direct isometry of $\mathbb{S}^2$ and has exactly two fixed points on the sphere. The group G has thus an invariant set of two elements. By considering germs in the neighborhood of these two points, one gets that G can be embedded into $2\cdot\mathrm{Diff}(\mathbb{R}^2,0)$ (2) and thus into $2\cdot\mathrm{Diff}(\mathbb{C}^2,0)$:

$$1\longrightarrow\mathrm{Diff}(\mathbb{C}^2,0)\longrightarrow 2\cdot\mathrm{Diff}(\mathbb{C}^2,0)\longrightarrow\mathbb{Z}/2\mathbb{Z}\longrightarrow 0.$$

Let us remark that $2\cdot\mathrm{Diff}(\mathbb{C}^2,0)$ is the projective limit of the algebraic groups $2\cdot\mathrm{Diff}_i$. The end of the proof is thus the same as the proof of Proposition 3.2 except that the subgroup of the semi-simple elements of $2\cdot\mathrm{Diff}_i$ embeds now in $2\cdot\mathrm{GL}(2,\mathbb{C})$; it is metabelian because it contains an abelian subgroup of index 2. □

Let Γ be a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$ for $n\geq 5$. Since Γ contains nilpotent subgroups of finite index of length $n-1$ (for example the group of upper triangular unipotent matrices) which are not metabelian one gets the following statement.

Corollary 4.3 **([10]).** — *Let Γ be a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$; as soon as $n\geq 5$ there is no embedding of Γ into $\mathrm{Diff}^\omega(\mathbb{S}^2)$.*

5. Nilpotent subgroups of length 3 of the group of analytic diffeomorphisms of $\mathbb{S}^2$

Let us precise Proposition 3.1 for nilpotent subalgebras of length 3 of $\widehat{\chi}(\mathbb{C}^2,0)$. Let $\mathfrak{l}$ be such an algebra. The center $Z(\mathfrak{l})$ of $\mathfrak{l}$ generates a subspace of dimension at most 1 of $\widehat{\chi}(\mathbb{C}^2,0)\otimes\widehat{K}(\mathbb{C}^2)$, for else $\mathfrak{l}$ would be abelian (Proposition 3.1) and this is impossible under our assumptions. So let us assume that the dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2,0)\otimes\widehat{K}(\mathbb{C}^2)$ is 1. There exists an element X in $Z(\mathfrak{l})$ with the following property: any element of $Z(\mathfrak{l})$ can be written uX with u in $\widehat{K}(\mathbb{C}^2)$. Let $\mathfrak{g}$ denote the abelian ideal of $\mathfrak{l}$ defined by

$$\mathfrak{g}=\big\{\widetilde{X}\in\mathfrak{l}\,\big|\,\exists u\in\widehat{K}(\mathbb{C}^2),\,\widetilde{X}=uX\big\}.$$

By hypothesis $\mathfrak{l}$ is not abelian. Let Y be in $\mathfrak{l}$; assume that its projection onto $\mathfrak{l}/\mathfrak{g}$ is a non trivial element of $Z(\mathfrak{l}/\mathfrak{g})$. Any vector field of $\mathfrak{l}$ can be written

$$uX+vY,\qquad\qquad u,v\in\widehat{K}(\mathbb{C}^2).$$

Since X, resp. Y belongs to $Z(\mathfrak{l})$, resp. $Z(\mathfrak{l}/\mathfrak{g})$ and since the length of $\mathfrak{l}$ is 3, one has

$$X(u)=Y^3(u)=X(v)=Y(v)=0. \tag{5.1}$$

If X and Y are non singular, one can choose formal coordinates x and y such that $X=\frac{\partial}{\partial x}$ and $Y=\frac{\partial}{\partial y}$. The previous conditions can be thus translated as follows: v is a constant and u is a polynomial in y of degree 2. We will see that we have a similar property without assumption on X and Y.

2. Let G be a group and let q be a positive integer; $q\cdot\mathrm{G}$ denotes the semi-direct product of G^q by $\mathbb{Z}/q\mathbb{Z}$ under the action of the cyclic permutation of the factors.

Lemma 5.1. — *Let X and Y be two vector fields of $\widehat{\chi}(\mathbb{C}^2,0)$ that commute and are not colinear. One can assume that $(X,Y)=\left(\frac{\partial}{\partial\widetilde{x}},\frac{\partial}{\partial\widetilde{y}}\right)$ where $\widetilde{x}$ and $\widetilde{y}$ are two independent variables in a Liouvillian extension of $\widehat{K}(\mathbb{C}^2,0)$.*

Proof. — Since X and Y are non colinear, there exist two 1-forms α, β with coefficients in $\widehat{K}(\mathbb{C}^2)$ such that

$$\alpha(X)=1, \qquad \alpha(Y)=0, \qquad \beta(X)=0, \qquad \beta(X)=1.$$

The vector fields X and Y commute if and only if α and β are closed (this statement of linear algebra is true for convergent meromorphic vector fields and is also true in the completion). The 1-form α is closed so according to [6] one has

$$\alpha=\sum_{i=1}^{r}\lambda_i\frac{d\widehat{\phi}_i}{\widehat{\phi}_i}+d\left(\frac{\widehat{\psi}_1}{\widehat{\psi}_2}\right)=d\left(\sum_{i=1}^{r}\lambda_i\log\widehat{\phi}_i+\frac{\widehat{\psi}_1}{\widehat{\psi}_2}\right)$$

where $\widehat{\psi}_1$, $\widehat{\psi}_2$ and the $\widehat{\phi}_i$ denote some formal series and the λ_i some complex numbers. One has a similar expression for β. So there exists a Liouvillian extension κ of $\widehat{K}(\mathbb{C}^2)$ having two elements $\widetilde{x}$ and $\widetilde{y}$ with $\alpha=d\widetilde{x}$ and $\beta=d\widetilde{y}$. One thus has

$$X(\widetilde{x})=1, \qquad X(\widetilde{y})=0, \qquad Y(\widetilde{x})=0, \qquad Y(\widetilde{y})=1.$$

$\square$

From (5.1) one gets: v is a constant and u is a polynomial in $\widetilde{y}$ of degree 2; so one proves the following statement.

Proposition 5.2. — *Let $\mathfrak{l}$ be a nilpotent subalgebra of $\widehat{\chi}(\mathbb{C}^2,0)$ of length 3. Then $\mathfrak{l}$ is isomorphic to a subalgebra of*

$$\mathfrak{n}=\left\{P(\widetilde{y})\frac{\partial}{\partial\widetilde{x}}+\alpha\frac{\partial}{\partial\widetilde{y}}\,\middle|\,\alpha\in\mathbb{C},\,P\in\mathbb{C}[\widetilde{y}],\,\deg P=2\right\}.$$

Remark 5.3. — We use a real version of this statement whose proof is an adaptation of the previous one: a nilpotent subalgebra $\mathfrak{l}$ of length 3 of $\widehat{\chi}(\mathbb{R}^2,0)$ is isomorphic to a subalgebra of

$$\mathfrak{n}=\left\{P(\widetilde{y})\frac{\partial}{\partial\widetilde{x}}+\alpha\frac{\partial}{\partial\widetilde{y}}\,\middle|\,\alpha\in\mathbb{R},\,P\in\mathbb{R}[\widetilde{y}],\,\deg P=2\right\}.$$

Theorem 5.4. — *Let Γ be a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$; as soon as $n\geq 4$ there is no embedding of Γ into $\mathrm{Diff}^\omega(\mathbb{S}^2)$.*

Proof. — Let $\mathrm{U}(4,\mathbb{Z})$ (resp. $\mathrm{U}(4,\mathbb{R})$) be the subgroup of unipotent upper triangular matrices of $\mathrm{SL}(4,\mathbb{Z})$ (resp. $\mathrm{SL}(4,\mathbb{R})$); it is a nilpotent subgroup of length 3. Assume that there exists an embedding from a subgroup Γ of finite index of $\mathrm{SL}(4,\mathbb{Z})$ into $\mathrm{Diff}^\omega(\mathbb{S}^2)$. Up to finite index Γ contains $\mathrm{U}(4,\mathbb{Z})$. Let us set $\mathrm{H}=\rho(\mathrm{U}(4,\mathbb{Z}))$. Up to finite index H has a fixed point (Proposition 4.1). One can thus see H as a subgroup of $\mathrm{Diff}(\mathbb{R}^2,0)\subset\widehat{\mathrm{Diff}}(\mathbb{R}^2,0)$ up to finite index.

Let us denote j^1 the morphism from $\widehat{\mathrm{Diff}}(\mathbb{R}^2,0)$ to Diff_i. Up to conjugation $j^1(\rho(\mathrm{U}(4,\mathbb{Z})))$ is a subgroup of

$$\left\{\begin{bmatrix}\lambda & t\\ 0 & \lambda\end{bmatrix}\,\middle|\,\lambda\in\mathbb{R}^*,\,t\in\mathbb{R}\right\}.$$

Up to index 2 one can thus assume that $j^1 \circ \rho$ takes values in the connected, simply connected group T defined by

$$\mathrm{T} = \left\{ \begin{bmatrix} \lambda & t \\ 0 & \lambda \end{bmatrix} \,\middle|\, \lambda, t \in \mathbb{R}, \lambda > 0 \right\}.$$

Let us set

$$\mathrm{Diff}_i(\mathrm{T}) = \left\{ f \in \mathrm{Diff}_i \mid j^1(f) \in \mathrm{T} \right\};$$

the group $\mathrm{Diff}_i(\mathrm{T})$ is a connected, simply connected, nilpotent and algebraic group. The morphism

$$\rho_i \colon \mathrm{U}(4,\mathbb{Z}) \to \mathrm{Diff}_i$$

can be extended to a unique continuous morphism $\widetilde{\rho}_i \colon \mathrm{U}(4,\mathbb{R}) \to \mathrm{Diff}_i(\mathrm{T})$ (*see* [**13, 12**]) so to an algebraic morphism [3]. Let us note that $\widetilde{\rho}_i(\mathrm{U}(4,\mathbb{Z}))$ is an algebraic subgroup of $\mathrm{Diff}_i(\mathrm{T})$ which contains $\rho_i(\mathrm{U}(4,\mathbb{Z}))$; in particular $\overline{\mathrm{H}_i} = \overline{\rho_i(\mathrm{U}(4,\mathbb{Z}))} \subset \widetilde{\rho}_i(\mathrm{U}(4,\mathbb{R}))$. By construction the family $(\mathrm{H}_i)_i$ is filtered; since the extension is unique, the family $(\widetilde{\rho}_i)_i$ is also filtered. Therefore $\mathrm{K} = \lim\limits_{\leftarrow} \overline{\mathrm{H}_i}$ is well defined. Since ρ is injective, H is a nilpotent subgroup of length 3; as $\mathrm{H} \subset \mathrm{K}$ and as any $\overline{\mathrm{H}_i}$ is nilpotent of length at most 3 the group K is nilpotent of length at most 3. For i sufficiently large $\widetilde{\rho}_i(\mathrm{U}(4,\mathbb{R}))$ is nilpotent of length 3; this group is connected so its Lie algebra is also nilpotent of length 3. Therefore the image of

$$D\widetilde{\rho} := \lim_{\leftarrow} D\widetilde{\rho}_i \colon \mathfrak{u}(4,\mathbb{R}) \to \widehat{\chi}(\mathbb{R}^2, 0)$$

is isomorphic to $\mathfrak{n}$ (Proposition 5.2). So there exists a surjective map ψ from $\mathfrak{u}(4,\mathbb{R})$ onto $\mathfrak{n}$. The kernel of ψ is an ideal of $\mathfrak{u}(4,\mathbb{R})$ of dimension 2; hence $\ker\psi = \langle \delta_{14}, a\delta_{13} + b\delta_{24} \rangle$ where the δ_{ij} denote the Kronecker matrices. One concludes by remarking that $\dim Z(\mathfrak{u}(4,\mathbb{R})/\ker\psi) = 2$ whereas $\dim Z(\mathfrak{n}) = 1$. $\square$

Corollary 5.5. — *The image of a morphism from a subgroup of* $\mathrm{SL}(n,\mathbb{Z})$ *of finite index to* $\mathrm{Diff}^\omega(\mathbb{S}^2)$ *is finite as soon as* $n \geq 4$.

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